Perturbation theory for coupled nonlinear Schrödinger equations

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We extend the perturbation theory of the nonlinear Schrödinger equation to the case of the integrable vector nonlinear Schrödinger equation. By applying the perturbed inverse scattering transform, we derive a set of nonlinear coupled evolution equations for the adiabatic change of the parameters of a vector soliton, in the presence of a generic perturbation. We show that the same equations may also be obtained by means of a Lagrangian variational approach. [S1063-651X(96)00811-2]

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I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation is a paradigm equation for describing wave propagation in weakly nonlinear and dispersive media [1]. A basic property of this equation is its complete integrability by means of the inverse scattering transform (IST) method [1–3]. The importance of the IST method is that it still permits one to analyze, by means of perturbative approaches, practical situations where the wave propagation is subject to small perturbations which break integrability. A perturbation theory for the NLS equation, valid for a generic conservative or dissipative perturbation, was developed with the help of the IST method [4,5].

On the other hand, a general perturbation theory based on the IST method for the integrable vector NLS equation [6], which describes the nonlinear coupling between different waves, has not been presented yet. Applications of the vector NLS equation range from plasma physics (e.g., coupling of Langmuir and transverse or sonic waves [7]) to nonlinear optics (e.g., coupling between orthogonal polarizations in a diffractive or dispersive dielectric [8]) and long-distance soliton-based communications [9,10]. Several perturbative analyses discussed the effect of specific conservative or dissipative perturbations on the adiabatic variations of the parameters of vector optical solitons [11]. In fact, the dynamics of the one-soliton parameters may be derived in different ways. For example, by means of the variation of the conserved quantities associated with the vector NLS equation, or by Lagrangian variational methods.

In this work, we derive the perturbation equations for the vector soliton parameters in the presence of a generic perturbation. In order to do that, we generalize the perturbed IST method [12–14] to the case of the vector NLS equation. We then show that the same results may equivalently be obtained by extending the Lagrangian perturbation method of Ref. [15]. Finally, we present an example of application to nonlinear fiber optics, namely, we derive perturbation equations for the passive mode locking of a vector soliton propagating in a ring fiber laser [16,17].

The present perturbation theory may be further extended to include the contribution of radiation, and to analyze complex phenomena such as inelastic soliton collisions or the generation of polarization *shadows* [18–20].

II. PERTURBATION EQUATIONS

We intend to formulate one-soliton first-order perturbation theory for the perturbed integrable vector nonlinear Schrödinger equation, that reads in dimensionless units as [6]

$$i\frac{\partial U}{\partial Z}+\frac{1}{2}\frac{\partial^2 U}{\partial T^2}+[|U|^2+|V|^2]U=iR_U, \tag{1}$$

$$i\frac{\partial V}{\partial Z}+\frac{1}{2}\frac{\partial^2 V}{\partial T^2}+[|U|^2+|V|^2]V=iR_{V.} \tag{2}$$

Here U and V represent, for example, the complex envelopes of the two orthogonal polarizations of a transverse electromagnetic field in a cubic nonlinear medium. We assume that the perturbation terms R_U, R_V are relatively small, and that they decay to zero sufficiently rapidly at infinity.

Whenever $R_U=R_V=0$, the one-soliton solution of Eqs. (1) and (2) is [6]

$$U_0(T,Z)=2\nu\cos(\theta)\text{sech}[2\nu(T-\xi)]e^{i[2\mu(T-\xi)+\delta_U]}, \tag{3}$$

$$V_0(T,Z)=2\nu\sin(\theta)\text{sech}[2\nu(T-\xi)]e^{i[2\mu(T-\xi)+\delta_V]}, \tag{4}$$

where $2\nu,\xi$, and 2μ represent the soliton amplitude, position, and frequency, whereas δ_U, δ_V are the phases of the orthogonal polarization components and θ is the polarization angle.

The single-soliton solution (3) and (4) depends on six parameters. The aim of the present paper is to show that, in the presence of generic perturbations, these parameters evolve slowly with the distance Z, according to the following set of coupled equations:

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$$\frac{d\nu}{dZ}=\frac{1}{2}\mathrm{Re}\int\left[\cos(\theta)R_Ue^{-i\beta_U}+\sin(\theta)R_Ve^{-i\beta_V}\right]\frac{1}{\cosh(x)}dx, \tag{5}$$

$$\frac{d\mu}{dZ}=\frac{1}{2}\mathrm{Im}\int\left[\cos(\theta)R_Ue^{-i\beta_U}+\sin(\theta)R_Ve^{-i\beta_V}\right]\frac{\tanh(x)}{\cosh(x)}dx, \tag{6}$$

$$\frac{d\theta}{dZ}=\frac{1}{4\nu}\mathrm{Re}\int\left[\cos(\theta)R_Ve^{-i\beta_V}-\sin(\theta)R_Ue^{-i\beta_U}\right]\frac{1}{\cosh(x)}dx, \tag{7}$$

$$\frac{d\xi}{dZ}=2\mu+\frac{1}{4\nu^2}\mathrm{Re}\int\left[\cos(\theta)R_Ue^{-i\beta_U}+\sin(\theta)R_Ve^{-i\beta_V}\right]\frac{x}{\cosh(x)}dx, \tag{8}$$

$$\frac{d\delta_U}{dZ}=2(\nu^2-\mu^2)+2\mu\frac{d\xi}{dZ}+\frac{1}{2\nu}\mathrm{Im}\int\left[\cos(\theta)R_Ue^{-i\beta_U}+\sin(\theta)R_Ve^{-i\beta_V}\right]\frac{1-x\tanh(x)}{\cosh(x)}dx$$
$$+\frac{1}{4\nu}\frac{\sin(\theta)}{\cos(\theta)}\mathrm{Im}\int\left[\sin(\theta)R_Ue^{-i\beta_U}-\cos(\theta)R_Ve^{-i\beta_V}\right]\frac{1}{\cosh(x)}dx, \tag{9}$$

$$\frac{d\delta_V}{dZ}=2(\nu^2-\mu^2)+2\mu\frac{d\xi}{dZ}+\frac{1}{2\nu}\mathrm{Im}\int\left[\cos(\theta)R_Ue^{-i\beta_U}+\sin(\theta)R_Ve^{-i\beta_V}\right]\frac{1-x\tanh(x)}{\cosh(x)}dx$$
$$-\frac{1}{4\nu}\frac{\cos(\theta)}{\sin(\theta)}\mathrm{Im}\int\left[\sin(\theta)R_Ue^{-i\beta_U}-\cos(\theta)R_Ve^{-i\beta_V}\right]\frac{1}{\cosh(x)}dx, \tag{10}$$

where $x=2\nu(T-\xi)$, $\beta_U=\mu/(\nu Z)+\delta_U$, and $\beta_V=\mu/(\nu Z)+\delta_V$. The derivation of Eqs. (5) and (10) will be outlined in the next two sections by means of two different approaches, that is, by means of the perturbed IST, and of the Lagrangian or variational method.

III. PERTURBED INVERSE SCATTERING TRANSFORM

A. Eigenvalue problem

As shown in Ref. [6], whenever $R_U=R_V\equiv 0$, the vector NLS equations (1) and (2) are exactly solvable by means of the IST method [2,3]. In fact, (1) and (2) may be derived from the compatibility condition of the eigenvalue problem

$$\frac{\partial}{\partial T}|f\rangle=\hat{M}|f\rangle,\quad \hat{M}=\begin{bmatrix}-i\lambda & iU^* & iV^*\\ iU & i\lambda & 0\\ iV & 0 & i\lambda\end{bmatrix}. \tag{11}$$

Let us consider the two sets of special solutions of (11), say $|\phi_i(T,\lambda)\rangle$ and $|\psi_i(T,\lambda)\rangle$ $(i=1,2,3)$, which are defined through their asymptotic behavior

$$|\phi_i\rangle_j=\delta_{ik}\exp\{-iI_j\lambda T\},\quad T\to-\infty,$$

$$|\psi_i\rangle_j=\delta_{ik}\exp\{-iI_j\lambda T\},\quad T\to+\infty,$$

with $I_1=1$ and $I_2=I_3=-1$. The kets $|\phi_i\rangle$, $|\psi_i\rangle$ are known as Jost functions: both represent a complete set of solutions to the eigenvalue problem (11). Hence we may express the elements of one set in terms of the other basis, for example,

$$|\phi_i(T,\lambda)\rangle=\sum_{j=1}^{3}\alpha_{ij}(\lambda)|\psi_i(T,\lambda)\rangle.$$

This expression defines the scattering matrix, say $\hat{\alpha}=\{\alpha_{i,j}\}$ $(i,j=1,2,3)$ of the system (11) in the basis of the Jost functions. Note that the entries of $\hat{\alpha}$ *do not depend on the time* T.

It has been previously shown [6] that the Jost functions $|\phi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ can be analytically continued into the upper half plane of the complex variable λ $\forall T$, whereas the same holds true in the lower half plane for $|\phi_2\rangle$, $|\phi_3\rangle$, and $|\psi_1\rangle$. Moreover, the analyticity of the Jost functions implies the analyticity of $\alpha_{11}(\lambda)$ in the region $\mathrm{Im}(\lambda)\geqslant 0$. Furthermore, let us denote by λ_k the $k=1,\ldots,N$ zeros of $\alpha_{11}(\lambda)$. These eigenvalues correspond to potentials U, V in Eqs. (11) which decay sufficiently rapidly as $|T|\to+\infty$. In correspondence with these eigenvalues, it turns out that

$$|\phi_1(T,\lambda_k)\rangle=\alpha_{12k}|\psi_2(T,\lambda_k)\rangle+\alpha_{13k}|\psi_1(T,\lambda_k)\rangle,$$
$$k=1,\ldots,N.$$

In the above expression α_{12k} and α_{13k} are two complex numbers which, if U and V are defined on compact support, are obtained by evaluating the scattering matrix elements on the eigenvalues, i.e., $\alpha_{12k}=\alpha_{12}(\lambda_k)$ and $\alpha_{13k}=\alpha_{13}(\lambda_k)$.

The main issue of the IST method is that the potentials U and V may be completely reconstructed at any Z, as long as one knows the evolution of their boundary values, residues, and discontinuities on the real axis: These quantities represent the set of scattering data [6], and read as

$$S_+=\left\{(\lambda_k,\gamma_{2k},\gamma_{3k})_{k=1}^{N},\frac{\alpha_{12}}{\alpha_{11}},\frac{\alpha_{13}}{\alpha_{11}}\right\},$$

where

$$\gamma_{nk}=\frac{\alpha_{1nk}}{\alpha'_{11k}}, \quad n=2,3, \quad k=1,\ldots,N.$$

Here, the prime denotes derivation with respect to λ and the subscript k indicates that the quantity is evaluated in correspondence with the kth zero λ_k of α_{11}.

Note that the γ_{nk} may be computed as

$$\gamma_{nk}=\lim_{\lambda\to\lambda_k}\gamma_n(\lambda)=\lim_{\lambda\to\lambda_k}(\lambda-\lambda_k)\frac{\alpha_{1n}(\lambda)}{\alpha_{11}(\lambda)}. \tag{12}$$

Hence the complete set of Z-evolution laws for the parameters characterizing an arbitrary solution of Eqs. (1) and (2) with $R_U=R_V\equiv 0$ may be derived from the spatial evolution of the set S_+. Whenever $R_U\neq 0$, and $R_V\neq 0$, the adiabatic (i.e., slow) evolution of the parameters of a solution of Eqs. (1) and (2) may still be found, as long as the space evolution of the scattering data can be determined. In the next paragraph, we illustrate the procedure that leads to the evolution of the scattering data in the presence of a perturbation.

B. Space evolution of scattering data

The scattering matrix elements α_{ik} associated with the eigenvalue problem (11) may be considered as functionals that depend on the field components U, V and their complex conjugates. Therefore their Z evolution may be expressed in the chain-rule form

$$\frac{d\alpha_{ij}}{dZ}=\int\left(\frac{\delta\alpha_{ij}(T')}{\delta U(T)}\frac{\partial U}{\partial Z}+\frac{\delta\alpha_{ij}(T')}{\delta U^*(T)}\frac{\partial U^*}{\partial Z}+\frac{\delta\alpha_{ij}(T')}{\delta V(T)}\frac{\partial V}{\partial Z}+\frac{\delta\alpha_{ij}(T')}{\delta V^*(T)}\frac{\partial V^*}{\partial Z}\right)dT. \tag{13}$$

Note that the instants of time T and T' are arbitrary, since α_{ij} does not depend on the time coordinate.

The variations of α_{ij} with respect to the potentials (U,U^*,V,V^*) may be computed by means of the equivalence [6]

$$\alpha_{ij}=\langle\psi_j|\phi_i\rangle=\sum_{n=1}^{3}\psi_j^{*(n)}\phi_i^{(n)},$$

which relates the α_{ik}'s to the Jost functions. Equation (3.2) yields, for instance,

$$\frac{\delta\alpha_{ij}(T')}{\delta U(T)}=\left\langle\frac{\delta\psi_j(T')}{\delta U(T)}\middle|\phi_i\right\rangle+\left\langle\psi_j\middle|\frac{\delta\phi_i(T')}{\delta U(T)}\right\rangle. \tag{14}$$

In order to evaluate the variational derivatives of the Jost functions, we use the fact that the right-hand side of Eq. (13) does not depend on time. Then, we may choose $T'>T$, so that $\delta\psi_j(T')/\delta U(T)=0$ since ψ_j is a plane wave that propagates to the left from $T\to+\infty$. As a consequence, for the causality principle the value $\psi_j(T')$ may not be affected by a variation of U which occurred at a time $T<T'$.

Hence it remains to evaluate the variations of the ϕ_i's with respect to the potentials (U,U^*,V,V^*). This may be done by evaluating the variation of the eigenvalue problem (11) and yields (a similar calculation was done for the scalar NLS equation in Refs. [1,4])

$$\frac{\partial}{\partial T}\frac{\delta|\phi_i(T')\rangle}{\delta U(T)}=\hat{M}\frac{\delta|\phi_i(T')\rangle}{\delta U(T)}+\frac{\delta\hat{M}}{\delta U(T)}|\phi_i(T')\rangle=\hat{M}\frac{\delta|\phi_i(T')\rangle}{\delta U(T)}+\begin{bmatrix}0&0&0\\ i&0&0\\ 0&0&0\end{bmatrix}\times\delta(T-T')|\phi_i(T')\rangle. \tag{15}$$

Since $|\phi_i(T')\rangle$ is defined in its asymptotic form as $T'\to-\infty$, the solution of (15) is uniquely determined by the additional condition $\delta\phi_i(T')/\delta U(T)=0$ for $T'<T$.

Now, we use again the fact that the choice of T and T' is arbitrary, and we set $T'=T+\epsilon$, $\epsilon\to 0$. Then, by integration of (15) over the interval $[T,T']$, Eq. (14) finally yields

$$\left|\frac{\delta\alpha_{ij}(T')}{\delta U(T)}\right\rangle=\left\langle\psi_j\middle|\begin{bmatrix}0&0&0\\ i&0&0\\ 0&0&0\end{bmatrix}\middle|\phi_i\right\rangle=i\phi_{i1}\psi_{j2}.$$

In a similar manner,

$$\left|\frac{\delta\alpha_{ij}(T')}{\delta U^*(T)}\right\rangle=i\phi_{i2}\psi_{j1},\quad\left|\frac{\delta\alpha_{ij}(T')}{\delta V(T)}\right\rangle=i\phi_{i1}\psi_{j3},$$

$$\left|\frac{\delta\alpha_{ij}(T')}{\delta V^*(T)}\right\rangle=i\phi_{i3}\psi_{j1},$$

so that

$$\frac{d\alpha_{11}}{dZ}=\int\left(i\frac{\partial U}{\partial Z}\phi_{11}\psi_{12}^*+i\frac{\partial U^*}{\partial Z}\phi_{12}\psi_{11}^*+i\frac{\partial V}{\partial Z}\phi_{11}\psi_{13}^*+i\frac{\partial V^*}{\partial Z}\phi_{13}\psi_{11}^*\right)dT, \tag{16}$$

$$\frac{d\alpha_{1n}}{dZ}=\int\left(i\frac{\partial U}{\partial Z}\phi_{11}\psi_{n2}^*+i\frac{\partial U^*}{\partial Z}\phi_{12}\psi_{n1}^*+i\frac{\partial V}{\partial Z}\phi_{11}\psi_{n3}^*+i\frac{\partial V^*}{\partial Z}\phi_{13}\psi_{n1}^*\right)dT \tag{17}$$

for $n=2,3$. Moreover, we may write

$$\frac{d}{dZ}\left(\frac{\alpha_{1n}}{\alpha_{11}}\right)=\frac{f_n(\phi,\psi)}{\alpha_{11}^2} \tag{18}$$

for $n=2$, 3, and with

$$f_n(\phi,\psi)=\int\Bigg[i\frac{\partial U}{\partial Z}\phi_{11}(\alpha_{11}\psi^*_{n2}-\alpha_{1n}\psi^*_{12})+i\frac{\partial U^*}{\partial Z}\phi_{12}(\alpha_{11}\psi^*_{n1}-\alpha_{1n}\psi^*_{11})\Bigg]+\Bigg[i\frac{\partial V}{\partial Z}\phi_{11}(\alpha_{11}\psi^*_{n3}-\alpha_{1n}\psi^*_{13})+i\frac{\partial V^*}{\partial Z}\phi_{13}(\alpha_{11}\psi^*_{n1}-\alpha_{1n}\psi^*_{11})\Bigg]dT.$$

In the following, we restrict our attention to the one-soliton case, i.e., to a single zero (which we denote as $\lambda=\lambda_1\equiv\lambda_S$) for α_{11}. We then may proceed as described by Newell for the case of the scalar NLS [12]. By using the definition of γ_n ($n=2,3$) given in (12), it is immediately verifiable that

$$\frac{d}{dZ}\left(\frac{\alpha_{1n}}{\alpha_{11}}\right)=\frac{1}{\lambda-\lambda_S}\frac{d\gamma_n}{dZ}+\frac{\gamma_n}{(\lambda-\lambda_S)^2}\frac{d\lambda_S}{dZ}.$$

On the other hand, by expanding in Taylor series (18) around $\lambda=\lambda_S$ one also obtains, at first order

$$\frac{d}{dZ}\left(\frac{\alpha_{1n}}{\alpha_{11}}\right)\simeq\frac{[f_{nS}+(\lambda-\lambda_S)f'_{nS}]}{[\alpha'_{11S}(\lambda-\lambda_S)]^2}\left[1-\frac{\alpha''_{11S}}{\alpha'_{11S}}(\lambda-\lambda_S)\right],$$

where, as usual, primes denote differentiation with respect to λ and the subscript S indicates that the quantity is evaluated in $\lambda=\lambda_S$. Finally, since

$$\frac{d\gamma_n}{dZ}\simeq\frac{d\gamma_{nS}}{dZ}-\gamma'_{nS}\frac{d\lambda_S}{dZ}+(\lambda-\lambda_S)\left(\frac{d\gamma'_{nS}}{dZ}-\gamma''_{nS}\frac{d\lambda_S}{dZ}\right)+\cdots$$

one finds

$$\frac{d\lambda_S}{dZ}=\frac{f_{nS}}{\gamma_{nS}(\alpha'_{11S})^2},\qquad(19)$$

and

$$\frac{d\gamma_{nS}}{dZ}=\frac{1}{(\alpha'_{11S})^2}\left[f'_{nS}-\frac{\alpha''_{11S}}{\alpha'_{11S}}\right].\qquad(20)$$

Equations (19) and (20) give the desired space evolution of the scattering data, which is required in order to completely characterize the solution of the initial problem.

C. Space evolution of single solitons

In the following, we specialize Eqs. (19) and (20) to the case of the one-soliton solution (3) and (4) of the perturbed nonlinear Schrödinger equations (1) and (2). In this case (see the Appendix for an outline of the calculation procedure), the Jost functions read as

$$|\phi_1\rangle=\frac{e^{-i\lambda T}}{\mu-\lambda-i\nu}\begin{bmatrix}i\nu\tanh(x)+\mu-\lambda\\ \nu\cos(\theta)\mathrm{sech}(x)e^{i\beta_U}\\ \nu\sin(\theta)\mathrm{sech}(x)e^{i\beta_V}\end{bmatrix},\qquad(21)$$

$$|\phi_2\rangle=\frac{e^{i\lambda T}}{\lambda-\mu-i\nu}\begin{bmatrix}\nu\cos(\theta)\mathrm{sech}(x)e^{-i\beta_U}\\ \lambda-\mu-i\nu+i\nu\cos^2(\theta)[\tanh(x)+1]\\ \frac{1}{2}i\nu\sin(2\theta)e^{i(\delta_V-\delta_U)}[\tanh(x)+1]\end{bmatrix},\qquad(22)$$

$$|\phi_3\rangle=\frac{e^{i\lambda T}}{\lambda-\mu-i\nu}\begin{bmatrix}\nu\sin(\theta)\mathrm{sech}(x)e^{-i\beta_V}\\ \frac{1}{2}i\nu\sin(2\theta)e^{-i(\delta_V-\delta_U)}[\tanh(x)+1]\\ \lambda-\mu-i\nu+i\nu\sin^2(\theta)[\tanh(x)+1]\end{bmatrix},\qquad(23)$$

and

$$|\psi_1\rangle=\frac{e^{-i\lambda T}}{\mu-\lambda+i\nu}\begin{bmatrix}i\nu\tanh(x)+\mu-\zeta\\ \nu\cos(\theta)\mathrm{sech}(x)e^{i\beta_U}\\ \nu\sin(\theta)\mathrm{sech}(x)e^{i\beta_V}\end{bmatrix},\qquad(24)$$

$$|\psi_2\rangle=\frac{e^{i\lambda T}}{\lambda-\mu+i\nu}\begin{bmatrix}\nu\cos(\theta)\mathrm{sech}(x)e^{-i\beta_U}\\ \lambda-\mu-i\nu+i\nu\cos^2(\theta)[\tanh(x)-1]\\ \frac{1}{2}i\nu\sin(2\theta)e^{i(\delta_V-\delta_U)}[\tanh(x)-1]\end{bmatrix},\qquad(25)$$

$$|\psi_3\rangle=\frac{e^{i\lambda T}}{\zeta-\mu+i\nu}\begin{bmatrix}\nu\sin(\theta)\mathrm{sech}(x)e^{-i\beta_V}\\ \frac{1}{2}i\nu\sin(2\theta)e^{-i(\delta_V-\delta_U)}[\tanh(x)-1]\\ \lambda-\mu-i\nu+i\nu\sin^2(\theta)[\tanh(x)-1]\end{bmatrix},\qquad(26)$$

with $x=2\nu(T-\xi)$. Therefore

$$\alpha_{11}(\lambda)=\frac{\lambda-\lambda_S}{\lambda-\lambda_S^*},$$

with $\lambda_S=\mu+i\nu$. Moreover, it may be verified that

$$\alpha_{12S}=i\cos(\theta)e^{i(\delta_U-2\lambda_S\xi)},$$

$$\alpha_{13S}=i\sin(\theta)e^{i(\delta_V-2\lambda_S\xi)},$$

so that

$$\gamma_{2S}=-2\nu\cos(\theta)e^{i(\delta_U-2\lambda_S\xi)},$$

$$\gamma_{3S}=-2\nu\sin(\theta)e^{i(\delta_V-2\lambda_S\xi)}.$$

By inserting the above eigenfunctions (21)–(26) in Eqs. (19) and (20), after some lengthy but straightforward algebra we obtain

$$\frac{d\lambda_S}{dZ}=\frac{i}{2}\mathrm{Re}\int[\cos(\theta)R_U e^{-i\beta_U}+\sin(\theta)R_V e^{-i\beta_v}]\frac{1}{\cosh(x)}dx+\frac{1}{2}\mathrm{Im}\int[\cos(\theta)R_U e^{-i\beta_U}+\sin(\theta)R_V e^{-i\beta_v}]\frac{\tanh(x)}{\cosh(x)}dx,$$

$$i\frac{d\delta_U}{dZ}-2i\lambda_S\frac{d\xi}{dZ}-\frac{\sin(\theta)}{\cos(\theta)}\frac{d\theta}{dZ}=\mathrm{Re}\int\left\{\left[z+\frac{\sin^2(\theta)}{2\cos^2(\theta)}\right]\frac{\cos(\theta)R_U e^{-i\beta_U}}{2\nu}+\left[z-\frac{1}{2}\right]\frac{\sin(\theta)R_V e^{-i\beta_V}}{2\nu}\right\}\frac{1}{\cosh(x)}dx$$
$$+i\mathrm{Im}\int\left\{\frac{\cos(\theta)}{2\nu}R_U e^{-i\beta_U}+\frac{\sin(\theta)}{2\nu}R_V e^{-i\beta_V}\right\}\frac{1-x\tanh(x)}{\cosh(x)}dx$$
$$+i\mathrm{Im}\int\left\{\frac{\sin^2(\theta)}{4\nu\cos(\theta)}R_U e^{-i\beta_U}-\frac{\sin(\theta)}{4\nu}\right\}\frac{1}{\cosh(x)}dx,$$

$$i\frac{d\delta_V}{dZ}-2i\lambda_S\frac{d\xi}{dZ}+\frac{\cos(\theta)}{\sin(\theta)}\frac{d\theta}{dZ}=\mathrm{Re}\int\left\{\left[z-\frac{1}{2}\right]\frac{\cos(\theta)R_U e^{-i\beta_U}}{2\nu}+\left[z+\frac{\cos^2(\theta)}{2\sin^2(\theta)}\right]\frac{\sin(\theta)R_V e^{-i\beta_V}}{2\nu}\right\}\frac{1}{\cosh(x)}dx$$
$$+i\mathrm{Im}\int\left\{\frac{\cos(\theta)}{2\nu}R_U e^{-i\beta_U}+\frac{\sin(\theta)}{2\nu}R_V \mathrm{e}^{-i\beta_V}\right\}\frac{1-x\tanh(x)}{\cosh(x)}dx$$
$$-i\mathrm{Im}\int\left\{\frac{\cos(\theta)}{4\nu}R_U e^{-i\beta_U}-\frac{\cos^2(\theta)}{4\nu\sin(\theta)}\right\}\frac{1}{\cosh(x)}dx.$$

The above equations readily lead to the desired evolution equations (5)–(10) for the soliton parameters.

IV. LAGRANGIAN METHOD

In this section we show that the same perturbation equations (5)–(10) for the evolution of the one-soliton parameters may also be derived by means of the Lagrangian method [14,15]. Note that this method does not require the unperturbed equations to be completely solvable by means of the IST. All that is necessary is simply that the unperturbed equations may be written in a Lagrangian form; additionally, a solitary wave solution should also clearly exist for the unperturbed problem. In the present case, it is easy to recast Eqs. (1) and (2) in the form

$$\frac{\delta\mathcal{L}_0}{\delta U^*}=iR_U,\quad \frac{\delta\mathcal{L}_0}{\delta V^*}=iR_V, \tag{27}$$

where the Lagrangian density is

$$\mathcal{L}_0=\mathrm{Im}\left[U\frac{\partial U^*}{\partial Z}+V\frac{\partial V^*}{\partial Z}\right]+\frac{1}{2}[|U|^2+|V|^2]^2 -\frac{1}{2}\left[\left|\frac{\partial U}{\partial T}\right|^2+\left|\frac{\partial U}{\partial T}\right|^2\right] \tag{28}$$

and the variational derivatives are defined as

$$\frac{\delta\mathcal{L}_0}{\delta U^*}=\sum_{n=0}^{+\infty}(-1)^n\frac{\partial^n}{\partial T^n}\left(\frac{\partial\mathcal{L}_0}{\partial^n U^*/\partial T^n}\right)-\frac{\partial}{\partial Z}\left(\frac{\partial\mathcal{L}_0}{\partial U^*/\partial z}\right), \tag{29}$$

$$\frac{\delta\mathcal{L}_0}{\delta V^*}=\sum_{n=0}^{+\infty}(-1)^n\frac{\partial^n}{\partial T^n}\left(\frac{\partial\mathcal{L}_0}{\partial^n V^*/\partial T^n}\right)-\frac{\partial}{\partial Z}\left(\frac{\partial\mathcal{L}_0}{\partial V^*/\partial z}\right). \tag{30}$$

The time-averaged [on the one-soliton solution $U=U_0$, $V=V_0$ (3) and (4)] Lagrangian reads as

$$L_0=\int\mathcal{L}_0[U_0,U_0^*,V_0,V_0^*]dT=8\mu\nu\frac{d\xi}{dZ}+\frac{8}{3}\nu^3-8\nu\mu^2 -4\nu\left[\cos^2(\theta)\frac{d\delta_U}{dZ}+\sin^2(\theta)\frac{d\delta_V}{dZ}\right]. \tag{31}$$

Let us now evaluate the variations of the Lagrangian L_0 with respect to the soliton parameters. These variations may be obtained in two different ways. First, one may directly use Eq. (31), which yields

$$\frac{\delta L_0}{\delta\mu}\equiv\frac{\partial L_0}{\partial\mu}-\frac{d}{dZ}\frac{\delta L_0}{\partial\mu/\partial Z}=8\nu\frac{d\xi}{dZ}-16\mu\nu, \tag{32}$$

$$\frac{\delta L_0}{\delta\nu}=8\mu\frac{d\xi}{dZ}+8(\nu^2-\mu^2) -4\nu\left[\cos^2(\theta)\frac{d\delta_U}{dZ}+\sin^2(\theta)\frac{d\delta_V}{dZ}\right], \tag{33}$$

$$\frac{\delta L_0}{\delta\xi}=-8\left(\mu\frac{d\nu}{dZ}+\nu\frac{d\mu}{dZ}\right), \tag{34}$$

$$\frac{\delta L_0}{\delta\delta_U}=4\cos^2(\theta)\frac{d\nu}{dZ}-8\nu\sin(\theta)\cos(\theta)\frac{d\theta}{dZ}, \tag{35}$$

$$\frac{\delta L_0}{\delta\delta_V}=4\sin^2(\theta)\frac{d\nu}{dZ}+8\nu\sin(\theta)\cos(\theta)\frac{d\theta}{dZ}, \tag{36}$$

$$\frac{\delta L_0}{\delta\theta}=8\nu\sin(\theta)\cos(\theta)\left(\frac{d\delta_U}{dZ}-\frac{d\delta_V}{dZ}\right). \tag{37}$$

On the other hand, the variations of L_0 may also be written in the chain-rule form, e.g.,

$$\frac{\delta L_0}{\delta \nu}=\int \left(\frac{\delta \mathcal{L}_0}{\delta U_0}\frac{\partial U_0}{\partial \nu}+\frac{\delta \mathcal{L}_0}{\delta U_0^*}\frac{\partial U_0^*}{\partial \nu}+\frac{\delta \mathcal{L}_0}{\delta V_0}\frac{\partial V_0}{\partial \nu} + \frac{\delta \mathcal{L}_0}{\delta V_0^*}\frac{\partial V_0^*}{\partial \nu}\right) dT.$$

By inserting in the above expression Eqs. (27) and the derivatives of the one-soliton solution (3) and (4), one obtains

$$\frac{\delta L_0}{\delta \mu}=\frac{2}{\nu}\mathrm{Re}\int [\cos(\theta)R_U e^{-i\beta_U}+\sin(\theta)e^{-i\beta_V}]\frac{x}{\cosh(x)}dx, \tag{38}$$

$$\frac{\delta L_0}{\delta \nu}=-\frac{2}{\nu}\mathrm{Im}\int [\cos(\theta)R_U e^{-i\beta_U} +\sin(\theta)e^{-i\beta_V}]\frac{1-x\tanh(x)}{\cosh(x)}dx, \tag{39}$$

$$\frac{\delta L_0}{\delta \xi}=-4\nu\ \mathrm{Im}\int [\cos(\theta)R_U e^{-i\beta_U}+\sin(\theta)e^{-i\beta_V}] \times\frac{\tanh(x)}{\cosh(x)}dx-4\mu\ \mathrm{Re}\int [\cos(\theta)R_U e^{-i\beta_U} +\sin(\theta)e^{-i\beta_V}]\frac{1}{\cosh(x)}dx, \tag{40}$$

$$\frac{\delta L_0}{\delta \delta_U}=2\mathrm{Re}\int \cos(\theta)R_U e^{-i\beta_U}\frac{1}{\cosh(x)}dx, \tag{41}$$

$$\frac{\delta L_0}{\delta \delta_V}=2\mathrm{Re}\int \sin(\theta)R_V e^{-i\beta_V}\frac{1}{\cosh(x)}dx, \tag{42}$$

$$\frac{\delta L_0}{\delta \theta}=2\ \mathrm{Im}\int [\sin(\theta)R_U e^{-i\beta_U} -\cos(\theta)R_V e^{-i\beta_V}]\frac{1}{\cosh(x)}dx. \tag{43}$$

By comparing the right-hand sides of (32)–(37) and Eqs. (38)–(43), one obtains again the perturbation equations (5) and (10).

V. POLARIZATION MODE LOCKING

As a specific example of application of the perturbation theory for vector NLS solitons, we outline the case of polarized solitons in fiber lasers [16,17]. In a fiber laser, linear and nonlinear birefringence in the optical fiber provide conservative or Hamiltonian perturbations to the vector solitons, whereas the gain medium, filters, and polarizing elements introduce dissipative perturbations. The stable propagation of vector solitons with a state of polarization which reproduces itself after each round-trip through the cavity results from the proper balance between all these perturbations. As soon as the field in the cavity has organized itself in the form of a stable soliton, the action of all the elements in each round-trip introduces only small changes to the soliton parameters. As a consequence, the soliton stability may be analyzed in terms of a pair of perturbed NLS equations which represent the averaging of the various elements over each pass. In a rather general form, we may write these equations as Eqs. (1) and (2) with

$$R_U=(\alpha+\delta+i\rho)U+(\gamma+i\kappa)V+\beta\frac{\partial^2 U}{\partial T^2}+i\sigma|V|^2U,$$
$$R_V=(\alpha-\delta-i\rho)V+(\gamma+i\kappa)U+\beta\frac{\partial^2 V}{\partial T^2}+i\sigma|U|^2V. \tag{44}$$

The meaning of the various perturbing terms in the right-hand side of Eqs. (44) is the following: $\alpha>0$ is the isotropic gain coefficient, whereas δ, γ, and β represent gain dichroism and dispersion. Moreover, ρ is the linear birefringence, κ is the linear coupling, and σ the differential cross-phase modulation coefficient which depends on the tensorial properties of the third-order susceptibility: in birefringent fibers, $\sigma=-1/3$.

By applying the general perturbation theory results (5)–(10) to the above case (44), one obtains the following equations for the adiabatic evolution of the vector soliton parameters:

$$\frac{d\nu}{dZ}=2\nu[\alpha+\delta\cos(2\theta)]-8\beta\nu\left(\frac{\nu^2}{3}+\mu^2\right) +2\gamma\nu\cos(\psi)\sin(2\theta),$$
$$\frac{d\mu}{dZ}=-\frac{16}{3}\beta\mu\nu^2,$$
$$\frac{d\xi}{dZ}=2\mu, \tag{45}$$
$$\frac{d\theta}{dZ}=-\kappa\sin(\psi)-\delta\sin(2\theta)+\gamma\cos(\psi)\cos(2\theta),$$
$$\frac{d\psi}{dZ}=2\rho-2\kappa\cos(\psi)\cot(2\theta)-\tfrac{8}{3}\sigma\nu^2\cos(2\theta) -2\gamma\csc(2\theta)\sin(\psi),$$

where $\psi=\delta_U-\delta_V$. The availability of a set of ordinary differential equations such as Eqs. (45) permits one to derive the conditions for the mode locking eigenstates of the laser: these are represented by the stable fixed points of the above system (45) (i.e., where $d\nu/dZ=d\mu/dZ=\cdots=d\psi/dZ=0$).

Clearly the limits of validity of soliton perturbation theory should be verified by comparing their predictions with the numerical solutions of the original Eqs. (1), (2), and (44). We do not intend to investigate here in detail these limits. Instead, we simply present in Figs. 1 and 2 two numerical examples which show that, even in the presence of relatively large perturbations, the perturbation theory is able to reproduce (over a limited distance) the numerically observed behavior for the polarization rotation of vector solitons. The solid curves in Figs. 1 and 2 illustrate the evolution with distance Z of the polarization angle θ and relative phase ψ from the perturbation theory, Eqs. (45). On the other hand, in these figures the dots show the exact numerical results which are obtained by extracting, through the direct spectral trans-

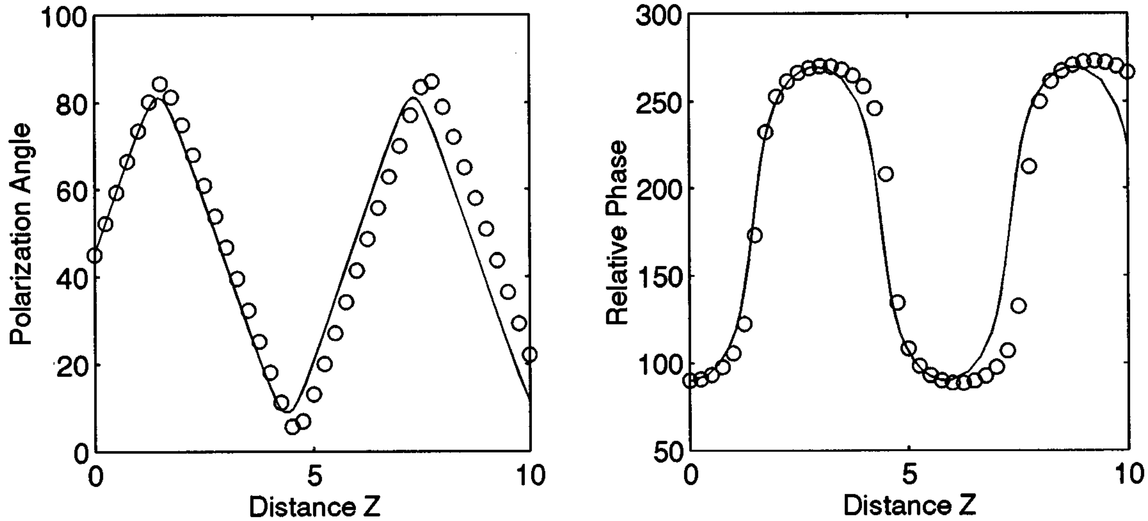

FIG. 1. Evolution of the polarization angle θ and relative phase ψ from perturbation theory (solid curves) and from the scattering transform of the numerical solutions (dots). Here the conservative perturbation is given by $\kappa=1/2$, $\sigma=-1/3$.

form, the two angles from successive profiles as they are computed from the numerical solutions (by a split-step Fourier method) of Eqs. (1), (2), and (44). In order to compute the direct spectral transform of the vector NLS equation from the numerical samples, we have extended the discrete scattering transform method of Ref. [21]. Figure 1 was obtained for a purely conservative perturbation to the vector soliton, i.e., we took a linear coupling $\kappa=1/2$ and a cross-phase modulation coefficient $\sigma=-1/3$. Moreover, we considered the initial values $\nu(0)=0.5$, $\theta(0)=\pi/4$, and $\psi(0)=\pi/2$. As can be seen, in this case there is a very good qualitative agreement between perturbation theory and numerical results. The quantitative agreement is also good over the first period of rotation of the soliton polarization. A slight mismatch of the soliton rotation period in the two cases introduces a progressively larger shift between perturbative and exact results for larger distances.

In the case of Fig. 2 we considered mixed conservative (same as in Fig. 1) and dissipative perturbations: we included the excess gain $\delta=0.1$ and the gain dispersion (or spectral filtering) coefficient $\beta=0.3$. Also, in this case $\nu(0)=0.6$. As can be seen, although the qualitative agreement of the perturbation results remains good, the mismatch of the soliton's period increases. We believe that this is largely due to the effect of radiation, which is generated as the soliton relaxes towards the fixed point of Eqs. (45) with amplitude $\nu=0.5$. In fact, Fig. 3 shows the fraction of soliton (solid curve) and radiation energy (dashed curve) as it is computed from the spectral transform of the numerical data. Note that the evolution of radiation in the perturbed vector NLS system may also be determined by extending the perturbed inverse scattering transform method which is presented here [19,20].

VI. CONCLUSIONS

We derived general evolution equations for the adiabatic evolution of the parameters of perturbed single solitons in the vector NLS equation. We have shown that these equa-

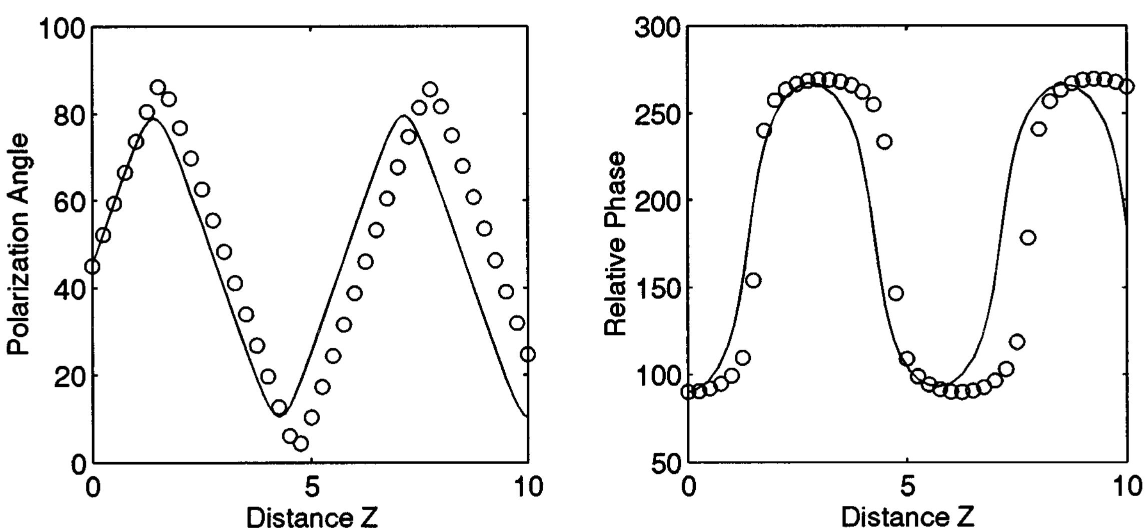

FIG. 2. Same as in Fig. 1, with mixed conservative and dissipative perturbations: here $\beta=3\delta=0.3$.

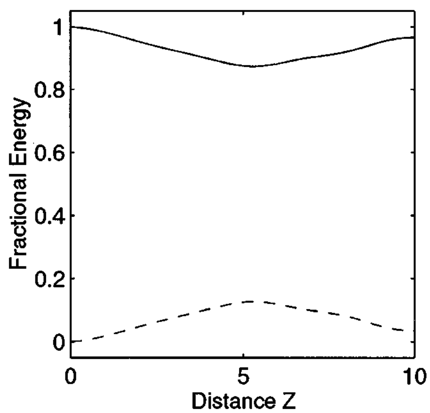

FIG. 3. Fraction of soliton (solid curve) and radiation (dashed curve) energies for the case of Fig. 2.

tions may equivalently be obtained by applying either the perturbed IST or the Lagrangian method. As an example of application of the theory, we briefly discussed the case of polarized solitons in optical fiber lasers.

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APPENDIX: EVALUATION OF THE JOST FUNCTIONS

Let us consider the eigensolutions of the scattering problem (11) that are associated with the one-soliton potential $U=U_0$, $V=V_0$, (3) and (4). We look for an eigenvector of the form

$$|f\rangle=e^{i\lambda T}|g\rangle. \tag{A1}$$

It follows that

$$i\frac{\partial g_1}{\partial T}+U_0^*g_2+V_0^*g_3=2\lambda g_1,$$
$$-i\frac{\partial g_2}{\partial T}-U_0g_1=0, \tag{A2}$$
$$-i\frac{\partial g_3}{\partial T}-V_0g_1=0.$$

With the ansatz

$$g_1=A_1\text{sech}[2\nu(T-\xi)]e^{i[2\mu(T-\xi)+\delta]} \tag{A3}$$

one obtains

$$g_2=iA_1\cos(\theta)e^{i(\delta_U-\delta)}\tanh[2\nu(T-\xi)]+B_2, \tag{A4a}$$

$$g_3=iA_1\sin(\theta)e^{i(\delta_V-\delta)}\tanh[2\nu(T-\xi)]+B_3. \tag{A4b}$$

Moreover, by using the first of Eqs. (A2), the following condition can be found:

$$A_1=\frac{\nu}{\lambda-\mu}[B_2\cos(\theta)^{i(\delta-\delta_U)}+B_3\sin(\theta)e^{i(\delta-\delta_V)}]. \tag{A5}$$

By imposing the boundary conditions

$$iA_1\cos(\theta)e^{i(\delta_U-\delta)}+B_2=1,$$
$$iA_1\sin(\theta)e^{i(\delta_V-\delta)}+B_3=0, \tag{A6}$$

which correspond to the Jost function ψ_2, one obtains from Eq. (A5)

$$A_1=\frac{\nu\cos(\theta)}{\lambda-\mu+i\nu}e^{i(\delta-\delta_U)},$$
$$B_2=\frac{\lambda-\mu+i\nu\sin^2(\theta)}{\lambda-\mu+i\nu}, \tag{A7}$$
$$B_3=\frac{-i\nu\sin(2\theta)}{2(\lambda-\mu+i\nu)}e^{i(\delta_V-\delta_U)}.$$

As a result, Eq. (25) is found.

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